

# $f(R)$ and $f(T)$ theories of modified gravity

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We briefly review  $f(R)$  theories, both in the metric and Palatini formulations, their scalar-tensor representations and the chameleon mechanism that could explain the absence of perceptible consequences in the Solar System. We also review  $f(T)$  theories, a different approach to modified gravity consisting in a deformation of the teleparallel equivalent of General Relativity. We show some applications to cosmology and cosmic strings. As  $f(R)$ 's,  $f(T)$  theories are not exempted from additional degrees of freedom; we also discuss this still open issue.

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## I. INTRODUCTION

In the last five decades many theories of modified gravity have been proposed in connection with different physical purposes. In the 60's, Brans and Dicke coupled a scalar field  $\phi$  to the metric  $g_{\mu\nu}$  to get a variable effective gravitational constant.<sup>1</sup> In Brans-Dicke theory, the scalar field is a new degree of freedom of the gravitational field, which is not directly coupled to the matter, but it exerts influence by entering the dynamical equations that govern the spacetime geometry. The gravitational Brans-Dicke action contains a new constant  $\omega$  that should be dictated by the experiment:

$$S_{BD}[g_{\mu\nu}, \phi] = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( \phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right), \quad (1)$$

where  $\kappa \equiv 8\pi G$ , and the signature  $+- --$  was adopted. Also in the 60's, the Einstein-Hilbert Lagrangian was added with terms quadratic in the curvature to tackle the renormalization of the theory.<sup>2</sup> In 1971 Lovelock<sup>3</sup> considered terms of higher order in the curvature as well; but he was driven by another motivation. While this kind of Lagrangians leads, in general, to fourth order dynamical equations because they contain second order derivatives, Lovelock obtained the more general Lagrangian polynomial in the curvature and leading to conserved second order equations for the metric. Lovelock discovered that the bigger the spacetime dimension is, the bigger is the number of terms these Lagrangians can contain. For instance, the Einstein-Lanczos Lagrangian,

$$L = -\frac{1}{2\kappa} (R + 2\Lambda) + \alpha (R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} + R^2 - 4R_{\mu\nu} R^{\mu\nu}), \quad (2)$$

is the Lovelock Lagrangian for dimensions 5 or 6; it is the Einstein-Hilbert Lagrangian with a cosmological constant plus a quadratic term. If the dimension is 4, then

the added quadratic term becomes a topological invariant (Euler's characteristic); so it does not contribute to the variation of the action. Then we recover the Einstein equations as the sole conserved second order equations for the metric in four dimensions.

In 1970 Buchdahl proposed to replace the Einstein-Hilbert scalar Lagrangian with a function of the scalar curvature, and studied its cosmological consequences.<sup>4</sup> This type of modified gravity is nowadays called a  $f(R)$  theory. In 1983 Milgrom<sup>5</sup> thought that the galactic rotation curves were an evidence of the fail of Newtonian gravity to describe gravitation in the weak field regime ( $a_g \ll a_o \approx 10^{-10} m s^{-2}$ ). According to Milgrom, no dark matter was needed to explain the data but a theory of modified gravity. In the deep *MOND* (Modified Newtonian Dynamics) regime of Milgrom's theory, the acceleration of gravity goes to  $a_g = \sqrt{a_o a_{gNewton}}$ . In the last decade Bekenstein<sup>6</sup> developed a relativistic theory of gravity named *TeVeS*, because it combines the metric tensor, a vector field and a scalar field. TeVeS includes Milgrom's weakening of Newtonian gravity in the weak field regime, and has also consequences for lensing phenomena, cosmology, etc.

String theory has been also a source of inspiration for theories of modified gravity. Just to mention a case, *DGP* gravity<sup>7</sup> describes the 4-dimensional universe as immersed in a 5-dimensional manifold. Thus a "normal" 5D gravity can cause large scale effects in 4D, as the accelerated expansion with no presence of dark energy. These 4D consequences are driven by a scalar field named *galileon* because of the symmetries it obeys.<sup>8,9</sup>

## II. $f(R)$ THEORIES

The simplest way of modifying Einstein's General Relativity is to replace the scalar Lagrangian  $R$  with a function  $f(R)$  of the scalar curvature:<sup>4,10,11</sup>

$$S = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R). \quad (3)$$

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By properly choosing the function  $f$ , one could generate “ $f(R)$ ” theories departing from General Relativity both at small and large scales. So, deformations at large curvatures could be employed for smoothing singularities; while deformations at large scales could be useful to geometrically explain the accelerated expansion without resorting to dark energy. The weak field regime of the deformed theory also opens a way to explain phenomena otherwise attributed to dark matter. References 12–18 are comprehensive reviews on  $f(R)$  theories.

As it happens in General Relativity, there are two ways of varying a metric theory of gravity. One can assume the Levi-Civita connection to write the Ricci tensor; so the metric is left as the sole dynamical variable. Alternatively, one could regard the affine connection  $\Gamma_{\mu\nu}^\lambda$  and the metric  $g_{\mu\nu}$  as independent dynamical variables. The first way of variation is called *metric formalism*; the second one is the *Palatini formalism*.<sup>19</sup> In General Relativity there is no difference between the results of metric and Palatini formalisms, since the affine connection making stationary the Einstein-Hilbert action is precisely the Levi-Civita connection. Actually, the equivalence between both formalisms is valid for all the Lovelock’s Lagrangians.<sup>20</sup> Instead, in  $f(R)$  theories both procedures should be separately studied because they yield different dynamics. Before choosing one of both formalisms, the variation of the  $f(R)$  Lagrangian density can be written as

$$\begin{aligned} \delta(f(R)\sqrt{-g}) &= f'(R)\sqrt{-g}\delta R + f(R)\delta\sqrt{-g} \\ &= f'(R)\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\ &+ \sqrt{-g}\left(f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\right)\delta g^{\mu\nu}, \end{aligned} \quad (4)$$

where the formula  $\delta\ln(\det[g_{\mu\nu}]) = -g_{\mu\nu}\delta g^{\mu\nu}$ , valid for any matrix, was used to vary the determinant of the metric. Besides,  $\delta R_{\mu\nu}$  can be expressed in terms of variations of the affine connection  $\Gamma_{\mu\nu}^\lambda$  (whatever  $\Gamma_{\mu\nu}^\lambda$  is; see for instance Ref. 21):

$$\delta R_{\mu\nu} = \nabla_\lambda \delta \Gamma_{\nu\mu}^\lambda - \nabla_\nu \delta \Gamma_{\lambda\mu}^\lambda. \quad (5)$$

Notice that the connection is not a tensor; but the difference  $\delta \Gamma_{\mu\nu}^\lambda$  between two different connections does transform as a tensor.

### III. METRIC FORMALISM FOR $f(R)$ THEORIES

If the Levi-Civita connection is assumed, then second derivatives of  $g_{\mu\nu}$  are under variation in Eq. (5). As a consequence, fourth-order Euler-Lagrange equations should be expected as a result of the (double) integration by parts induced by the variation (5). It is, however, remarkable that  $g^{\mu\nu}\delta R_{\mu\nu}$  is, in this case, a four-divergence. This is because the Levi-Civita connection is *metric*, so the metric can enter the covariant derivative. This is the reason why General Relativity ( $f'(R) = 1$ ) remains

as a theory governed by second order dynamical equations. Contrarily,  $f(R)$  theories in the metric formalism are characterized by fourth-order dynamical equations:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] f'(R) = \kappa T_{\mu\nu}. \quad (6)$$

Notice that  $f'(R)$  acts as renormalizing the gravitational constant  $\kappa$ ; so, only functions with  $f' > 0$  should be considered (besides,  $f'' > 0$  to avoid instabilities<sup>22–24</sup>). Differing from General Relativity, these equations link the scalar curvature  $R$  and the trace  $T$  of the energy-momentum tensor not algebraically but differentially. In fact, the trace of Eq. (6) is

$$f'(R)R - 2f(R) + 3\square f'(R) = \kappa T, \quad (7)$$

which displays the propagation of a new degree of freedom associated with  $f'(R)$  (this degree of freedom is absent in General Relativity since it is  $f'(R) = 1$ ).

A  $f(R)$  theory can be rephrased as a scalar-tensor theory governed by second order dynamical equations.<sup>25–27</sup> To show it, let us start from the following action containing a metric tensor  $g_{\mu\nu}$  and a scalar field  $\phi$ :

$$S_{grav}[g_{\mu\nu}, \phi] = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi R - V(\phi)]. \quad (8)$$

The variation with respect to  $\phi$  gives  $R = V'(\phi)$ , so linking the scalar field to the metric. This result also implies that the Lagrangian in (8) is nothing but the Legendre transform of the function  $V(\phi)$ ; therefore, it depends just on  $R$ . So we can call it  $f(R)$ :

$$f(R) \equiv \phi R - V(\phi). \quad (9)$$

By anti-transforming, one also gets

$$\phi = f'(R). \quad (10)$$

These results show that a  $f(R)$  theory in the metric formalism is dynamically equivalent to the action (8), where  $f(R)$  and  $V(\phi)$  are related through the Legendre transformation (9). Notice that  $S_{grav}[g, \phi]$  in Eq. (8) resembles a Brans-Dicke theory with  $\omega = 0$  (absence of kinetic term).

The action (8) is written in the so called *Jordan frame* representation of the theory. By transforming to the *Einstein frame* representation we will obtain second order dynamical equations. So let us define

$$\begin{aligned} \phi &\rightarrow \tilde{\phi} = \sqrt{\frac{3}{2\kappa}} \ln \phi \\ g_{\mu\nu} &\rightarrow \tilde{g}_{\mu\nu} = \phi g_{\mu\nu}, \quad \sqrt{-g} = \phi^{-2} \sqrt{-\tilde{g}}, \end{aligned} \quad (11)$$

where  $\tilde{g}_{\mu\nu}$  is conformally related to  $g_{\mu\nu}$  through the scalar field  $\phi$ . Then, one applies the relation for the scalar curvatures of conformally related metrics,<sup>28</sup>

$$\phi \tilde{R} = R - \frac{3}{2} g^{\mu\nu} \partial_\mu \ln \phi \partial_\nu \ln \phi - 3 \square \ln \phi, \quad (12)$$

and throws out a surface term to write the action in the form<sup>91</sup>

$$S'_{grav}[\tilde{g}_{\mu\nu}, \tilde{\phi}] = - \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} + U(\tilde{\phi}) \right] \quad (13)$$

where the potential is

$$U(\tilde{\phi}) = -\frac{V(\phi)}{2\kappa \phi^2} = \frac{f(R) - R f'(R)}{2\kappa f'(R)^2}. \quad (14)$$

As can be seen in Eq. (13), the action in the Einstein frame gets a canonical form: it describes a “gravitational field”  $\tilde{g}_{\mu\nu}$  and a minimally coupled scalar field  $\tilde{\phi}$ , governed by standard second order equations. So, we have success in reducing the order of the equations (of course, by widening the number of variables). An easy identification of the degrees of freedom is now possible: 2 degrees of freedom related to the tensor  $\tilde{g}_{\mu\nu}$  plus 1 degree of freedom associated with the massive scalar field  $\tilde{\phi}$ .

### A. The chameleon effect

Because of the observations in the Solar System, Brans-Dicke theory is constrained to values  $|\omega| > 40000$ . Despite that metric  $f(R)$  theories have  $\omega = 0$  in their equivalent representation (9), they are not ruled out. This is so because they contain a potential  $V(\phi)$  (or  $U(\tilde{\phi})$ ) that is absent in Brans-Dicke theory. This potential could be useful to hide the scalar degree of freedom within the Solar System. In other words, the observations in the Solar System could agree with metric  $f(R)$  theories, whenever the scalar degree of freedom does not appreciably distort the spherically symmetric outer static Schwarzschild solution for a typical stellar object. Even so, the scalar field could have physical effects at other scales, as to be the cause of the accelerated cosmological expansion, etc. This behavior, called the *chameleon effect*, has been proposed in Ref. 29 (cf. Refs. 16, 30–35), and is strongly dependent on the choice of the potential  $U$ .

The idea can be exemplified by considering solutions to the dynamical equation (7) for  $\phi = f'(R)$ . Alternatively, this equation can be also obtained by adding the action (13) with an action for matter minimally coupled to the metric  $g_{\mu\nu} = e^{(-\sqrt{2\kappa/3}\tilde{\phi})} \tilde{g}_{\mu\nu}$ , and varying with respect to  $\tilde{\phi}$ . We are interested in static spherically symmetric solutions. In such case, the resulting equation reduces to

$$\nabla^2 \tilde{\phi} = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\tilde{\phi}}{dr} \right) = U'(\tilde{\phi}) - \rho \sqrt{\frac{\kappa}{6}} \exp \left[ -\sqrt{\frac{8\kappa}{3}} \tilde{\phi} \right], \quad (15)$$

where  $\rho \equiv T_{mat}^\mu{}_\mu > 0$ . Here we are momentarily ignoring the back-reaction on the metric, by choosing a Minkowskian background  $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ .

We will divide the space in two regions of constant density: the region inner to a spherical star of radius  $R_\odot$

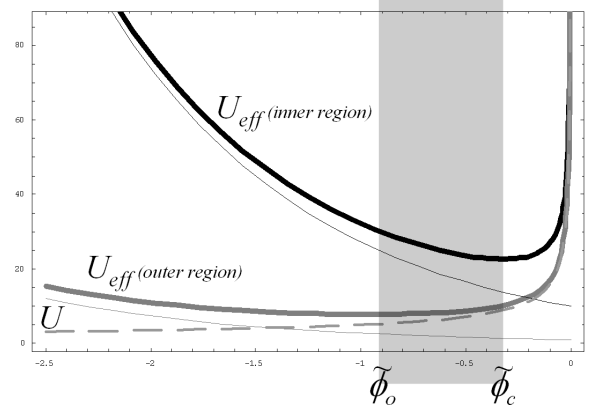


FIG. 1. Effective potentials in the inner and outer regions. The (chosen) potential  $U$  is the dashed line. The corresponding exponential terms in Eq. (16) are represented by the thin lines.

and density  $\rho_c$ , and the outer region filled by a medium of lower density  $\rho_o$ . For a constant energy density  $\rho$ , we can define (in each region) the effective potential

$$U_{eff} \equiv U(\tilde{\phi}) + \frac{\rho}{4} \exp \left[ -\sqrt{\frac{8\kappa}{3}} \tilde{\phi} \right]. \quad (16)$$

We have to choose a potential  $U$  allowing the chameleon effect to become apparent. In Figure 1,  $U$  has been chosen in such a way that  $U_{eff}$  has a minimum in both regions (we call them  $\tilde{\phi}_o$  and  $\tilde{\phi}_c$ ). We will search for a solution varying between  $\tilde{\phi}_c$  at the center of the star and  $\tilde{\phi}_o$  at infinity (grey strip in Figure 1). Outside the star, we expand  $U_{eff}$  at the minimum  $\tilde{\phi}_o$ :

$$U_{eff} \simeq m^2 (\tilde{\phi} - \tilde{\phi}_o)^2 / 2 + \text{constant}, \quad (17)$$

where  $m$  is the mass of the field in this approach. Thus, the outer solution is

$$\tilde{\phi} \simeq \tilde{\phi}_o + \frac{C}{r} \exp[-m(r - R_\odot)]. \quad (18)$$

Inside the star, we assume that the exponential term  $U_{eff}$  dominates on  $U$ . Then

$$\nabla^2 \tilde{\phi} \approx -\rho_c \sqrt{\frac{\kappa}{6}} \exp \left[ -\sqrt{\frac{8\kappa}{3}} \tilde{\phi} \right] \approx -\rho_c \sqrt{\frac{\kappa}{6}}, \quad (19)$$

if  $\sqrt{8\kappa/3} |\tilde{\phi}| \ll 1$ . So, the inner solution is

$$\tilde{\phi} \simeq \tilde{\phi}_c - \rho_c \sqrt{\frac{\kappa}{24}} R_s^2 \left( \frac{r^2}{3 R_s^2} + \frac{2 R_s}{3 r} - 1 \right). \quad (20)$$

Notice that the integration constant  $R_s$  fulfills

$$\tilde{\phi}(R_s) = \tilde{\phi}_c, \quad \tilde{\phi}'(R_s) = 0. \quad (21)$$

So, one can take Eq. (20) to be the inner solution for  $R_s < r < R_\odot$ , and extended it as the constant  $\tilde{\phi} = \tilde{\phi}_c$

to  $r = 0$  (in fact,  $U'_{eff}(\tilde{\phi}_c) = 0$ ). Thus,  $R_s$  in Eq. (20) and  $C$  in Eq. (18) remain as two integration constants to be determined by the continuity of the solution (20) and (18) and its derivative at  $r = R_\odot$ . By assuming that  $m R_\odot \ll 1$ , one obtains the following two equations for  $R_s$  and  $C$ :

$$\sqrt{\frac{3}{2\kappa}} \left[ 1 - \left( \frac{R_s}{R_\odot} \right)^2 \right] \simeq \epsilon, \quad (22)$$

$$C \simeq \sqrt{\frac{2}{3\kappa}} \Phi_N R_\odot \left[ 1 - \left( \frac{R_s}{R_\odot} \right)^3 \right], \quad (23)$$

where  $\Phi_N = \kappa \rho_c R_\odot^2 / 6$  is the Newtonian potential on the surface of the star, and

$$\epsilon \equiv \sqrt{\kappa} \frac{\tilde{\phi}_c - \tilde{\phi}_o}{\Phi_N}. \quad (24)$$

The chameleon effect happens when the potential  $U$  is such that  $\epsilon \ll 1$ . In fact, in such case it is

$$\sqrt{\kappa} C \approx \epsilon \Phi_N R_\odot \ll \kappa M_\odot, \quad (25)$$

so the effect of the potential  $\tilde{\phi}$  around the star is negligible compared with Newtonian gravity.<sup>92</sup> Besides,

$$\frac{R_\odot - R_s}{R_\odot} \approx \frac{\epsilon}{\sqrt{6}} \ll 1, \quad (26)$$

then the inner solution differs from  $\tilde{\phi}_c$  just in a thin-shell near the surface.

The back-reaction on the metric has been considered in Ref. 32; it is proved that the PPN parameter characterizing the departure from the Schwarzschild metric is  $\gamma \simeq 1 + \sqrt{2/3} \epsilon$ . The Cassini tracking constrains  $\epsilon$  to be  $\epsilon \lesssim 10^{-5}$  in the Solar System. Since the Newtonian potential on the Sun surface is  $\Phi_{Sun} \sim 10^{-6}$ , one obtains

$$\sqrt{\kappa} (\tilde{\phi}_{Sun} - \tilde{\phi}_o) \lesssim 10^{-11}. \quad (27)$$

The viable  $f(R)$  theories are those having a potential  $U$  accomplishing this relation. A typical  $f(R)$  used to model the accelerated expansion is:<sup>32,34,36,37</sup>

$$f(R) = R + \frac{\mu^{2(n+1)}}{R^n}, \quad (28)$$

because the  $R^{-n}$  term dominates at low curvature. By replacing the potential  $U$  of Eq. (14) in Eq. (16), one gets

$$U_{eff}(\tilde{\phi}) = -\frac{(n+1)\mu^2}{2\kappa\phi^2} \left( \frac{1-\phi}{n} \right)^{\frac{n}{n+1}} + \frac{\rho}{4\phi^2}, \quad (29)$$

where  $\phi = \exp[\sqrt{2\kappa/3}\tilde{\phi}]$ . If  $\rho\kappa \gg \mu^2$ , the minima of  $U_{eff}$  in each region are

$$\sqrt{2\kappa/3} \tilde{\phi}_{o,c} \simeq -n \left( \frac{\mu^2}{\rho_{o,c}\kappa} \right)^{n+1}, \quad (30)$$

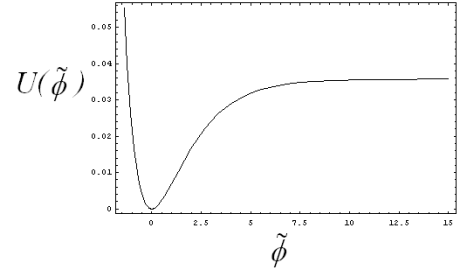


FIG. 2. Potential  $U(\tilde{\phi})$  in the model  $f(R) = R + \alpha R^2$ .

which are very near to zero as required by Eq. (27). This also means that there is no sensitive difference between metrics  $g_{\mu\nu}$  and  $\tilde{g}_{\mu\nu}$  (see Eq. (11)). In the Solar System, the density  $\rho_o$  outside the Sun should be replaced by the mean density of the barionic matter in the galaxy ( $\rho_{galaxy}\kappa$  is around  $10^5$  times the squared Hubble constant). Since  $\rho_{galaxy} \ll \rho_{Sun}$ , the Eq. (27) becomes

$$n \left( \frac{\mu^2}{\rho_{galaxy}\kappa} \right)^{n+1} \lesssim 10^{-11}, \quad (31)$$

in order that the model (28) be acceptable. The above used condition  $m R_\odot \ll 1$  can be accomplished too; in fact

$$m^2 = U''_{eff}(\tilde{\phi}_o) \sim \rho_{galaxy}\kappa \left( \frac{\rho_{galaxy}\kappa}{\mu^2} \right)^{n+1}. \quad (32)$$

## B. Metric $f(R)$ theories in cosmology

Within the framework of a FRW universe, it has been shown that the field  $\tilde{\phi}$  is attracted to the minimum of the potential (29), and then adiabatically evolves following the Eq. (30) with  $\rho = \rho_{universe}(t)$ . Using the usual approximations, it is obtained that the net effect of the presence of  $\tilde{\phi}$  is the adding of a constant to the density of matter.<sup>32,38</sup> In such case, the cosmological effects resulting from the model (28) would be undistinguishable from a mere cosmological constant (“vanilla dark energy”). The growing of inhomogeneities are, however, a more promising arena to distinguish among  $f(R)$  theories and the  $\Lambda$ CDM model.<sup>39–41</sup>

$f(R)$  theories have also been applied to modify the high curvature regime. The simplest example is

$$f(R) = R + \alpha R^2, \quad (33)$$

which produces inflation, with  $\tilde{\phi}$  playing the role of inflaton.<sup>42,43</sup> Figure 2 shows that the potential  $U(\tilde{\phi})$  is nearly flat for large values of  $\tilde{\phi}$ , as required to get inflation.

Other cosmological effects, such as lensing due to overdensities of matter, have also been considered in the framework of metric  $f(R)$  theories.<sup>44</sup>

#### IV. PALATINI FORMALISM FOR $f(R)$ THEORIES

In Palatini formalism<sup>19,45</sup> the connection  $\Gamma_{\mu\nu}^\lambda$  and the metric  $g_{\mu\nu}$  are regarded as dynamical variables to be independently varied. Thus,  $\nabla$  in Eq. (5) is just the covariant derivative for an arbitrary connection  $\Gamma_{\mu\nu}^\lambda$ . The variation of the action with respect to the connection involves the integration by parts of the first term in the Eq. (4); what results is not a dynamical equation but a constraint for the connection:<sup>30,46,47</sup>

$$\Gamma_{\mu\nu}^\lambda = \frac{g^{\lambda\sigma}}{2f'(R)} \left[ \partial_\mu (f'(R) g_{\nu\sigma}) + \partial_\nu (f'(R) g_{\mu\sigma}) - \partial_\sigma (f'(R) g_{\mu\nu}) \right] \quad (34)$$

(this is the result when torsion is neglected<sup>48,49</sup>). General Relativity is a special case, in the sense that the connection (34) is the Levi-Civita connection when  $f(R) = R$ ; so, no difference exists between metric and Palatini formalisms in General Relativity. But, in a general case, the connection (34) is not metric for  $g_{\mu\nu}$ , but for the conformal metric  $\tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu}$ .

On the other hand, the variation of the action with respect to the metric in Eq. (4) does yield dynamical equations:

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (35)$$

where  $R_{\mu\nu}$  and  $R$  are built with the connection (34). Here  $T_{\mu\nu}$  is the usual energy-momentum tensor, whenever the action for matter does not contain covariant derivatives. As a remarkable difference compared with the metric formalism, the trace of Eq. (35) does not govern the propagation of a scalar degree of freedom but it is a mere algebraic relation between the curvature  $R$  and the matter distribution:

$$f'(R) R - 2f(R) = \kappa T, \quad (36)$$

so suggesting that Palatini formalism does not harbor additional degrees of freedom, as confirmed by means of the Einstein frame representation of Palatini dynamics.<sup>15</sup> However, the relation  $R = R(T)$  in Eq. (36) implies that the connection (34) depends on first derivatives of  $T$ . Therefore, Eq. (35) involves second derivatives of  $T$ , which is a very unlike coupling between geometry and matter. This feature is the source of several troubles: juncture conditions on the surface of spherically symmetric bodies leading to curvature divergences even for reasonable state equations of matter<sup>52,53</sup> (however, see Ref. 54), incompatibilities with the stability of microscopic systems<sup>55-57</sup> and non-well formulated Cauchy problem unless the trace  $T$  is constant.<sup>58</sup>

#### V. $f(T)$ THEORIES

General Relativity can be reformulated in a *teleparallel* framework by taking the field of orthonormal frames or

*tetrads* as the dynamical variable instead of the metric tensor.<sup>59</sup> The tetrad is a basis  $\{\mathbf{e}_a(\mathbf{x})\}$ ,  $a = 0, 1, 2, 3$ , of vectors in the spacetime. Each vector  $\mathbf{e}_a$  can be decomposed in a coordinate basis, so giving the components  $e_a^\mu$ ; thus, the orthonormality condition reads:

$$\eta_{ab} = g_{\mu\nu} e_a^\mu e_b^\nu, \quad (37)$$

where  $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ . This relation can be inverted with the help of the co-frame  $\{\mathbf{e}^a\}$ , defined as

$$e_a^\mu e_\mu^b = \delta_a^b, \quad (38)$$

to obtain the metric starting from the tetrad:

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b \Rightarrow \sqrt{-g} = \det[e_\mu^a] \equiv e. \quad (39)$$

The Teleparallel Equivalent of General Relativity (TEGR) is a theory for the tetrad, whose dynamical equations are equivalent to Einstein equations whenever the tetrad is related to the metric through the Eq. (39). The TEGR Lagrangian does not contains second derivatives because it is quadratic in the tensor

$$T^\mu{}_{\nu\rho} = e_a^\mu (\partial_\nu e_\rho^a - \partial_\rho e_\nu^a), \quad (40)$$

which is reminiscent of the electromagnetic field tensor (in fact, it is built of the set of four exact 2-forms  $T^a \equiv de^a$ ). The tensor (40) can be regarded as the *torsion* of the Weitzenböck connection,

$$\Gamma_{\rho\nu}^\mu \equiv e_a^\mu \partial_\nu e_\rho^a = -e_\rho^a \partial_\nu e_a^\mu. \quad (41)$$

Weitzenböck spacetime has torsion but it is flat, because the Riemann tensor associated with the connection (41) is identically null. The connection (41) has the nice property that a vector is parallel-transported iff its projections on the tetrad remain constant; in fact,  $\nabla_\nu V^\mu = e_a^\mu \partial_\nu (e_\lambda^a V^\lambda)$ . Moreover, Weitzenböck connection is metric compatible since  $\nabla_\nu e_a^\mu \equiv 0$ . Weitzenböck connection could be compared with Levi-Civita connection by using Eq. (39). It results that they differ in a tensor named *contorsion*. The contorsion takes part in the TEGR Lagrangian, since the TEGR action is:<sup>60,61</sup>

$$S_T[e^a] = \frac{1}{2\kappa} \int d^4x e S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} \equiv \frac{1}{2\kappa} \int d^4x e \mathbb{S} \cdot \mathbb{T}, \quad (42)$$

where

$$2 S_\rho{}^{\mu\nu} \equiv \underbrace{(T_\rho{}^{\mu\nu} - T^{\mu\nu}{}_\rho + T^{\nu\mu}{}_\rho)}_{\text{contorsion } K^{\mu\nu}{}_\rho} + T_\lambda{}^{\lambda\mu} \delta_\rho^\nu - T_\lambda{}^{\lambda\nu} \delta_\rho^\mu. \quad (43)$$

The equivalence between TEGR action and Einstein-Hilbert action comes from the fact that their Lagrangians differ in a four-divergence:

$$-e R[e^a] = e \mathbb{S} \cdot \mathbb{T} - 2 \partial_\rho (e T^\mu{}_\mu{}^\rho), \quad (44)$$

where  $R[e^a]$  is the scalar curvature for the Levi-Civita connection, with the metric replaced with (39). In particular, the four-divergence encapsulates all the second derivatives contained in the Einstein-Hilbert Lagrangian.

In the same spirit than a  $f(R)$  theory, a  $f(T)$  theory consists in a deformation of the TEGR Lagrangian:<sup>62–66</sup>

$$S_T = \frac{1}{2\kappa} \int d^4x e \mathbb{S} \cdot \mathbb{T} \longrightarrow S = \frac{1}{2\kappa} \int d^4x e f(\mathbb{S} \cdot \mathbb{T}). \quad (45)$$

But, differing from  $f(R)$  theories, the dynamical equations in  $f(T)$  theories are always second order because the Lagrangian does not contain second derivatives. For matter coupled to the metric in the usual way, they are

$$\begin{aligned} & 4 \left[ e^{-1} \partial_\mu (e S_a^{\mu\nu}) + e_a^\lambda T_{\mu\lambda}^\rho S_\rho^{\mu\nu} \right] f'(\mathbb{S} \cdot \mathbb{T}) \\ & + 4 S_a^{\mu\nu} \partial_\mu (\mathbb{S} \cdot \mathbb{T}) f''(\mathbb{S} \cdot \mathbb{T}) - e_a^\nu f(\mathbb{S} \cdot \mathbb{T}) \\ & = -2\kappa e_a^\lambda T_{\lambda}^\nu, \end{aligned} \quad (46)$$

where  $T_{\lambda}^\nu$  is the energy-momentum tensor.

### A. Cosmology

The first  $f(T)$  model was proposed to avoid the Big-Bang singularity and obtain inflation without resorting to an inflaton.<sup>62</sup> But most of the cosmological applications concentrated in the late accelerated expansion of the universe.<sup>64,65,67–71</sup> A flat FRW universe is described by

$$\begin{aligned} e_\mu^a &= \text{diag}[1, a(t), a(t), a(t)] \text{ in comoving coordinates,} \\ \mathbb{S} \cdot \mathbb{T} &= -6 H^2, \end{aligned} \quad (47)$$

where  $H \equiv \dot{a}/a$  is the Hubble parameter. Thus, the dynamical equations (46) become

$$\begin{aligned} 12 H^2 f'(-6H^2) + f(-6H^2) &= 2\kappa \rho, \quad (48) \\ -4 \dot{H} f'(-6H^2) + 48 H^2 \dot{H} f''(-6H^2) &= 2\kappa (\rho + p), \end{aligned}$$

where  $\rho$  and  $p$  are the energy density and pressure of the fluid of matter (the conservation law  $\dot{\rho} = -3H(\rho + p)$  is guaranteed by Eq. (48)).

In Ref. 62 a high curvature deformation,  $f(T) = \lambda(\sqrt{1 + 2T/\lambda} - 1)$ , was proposed to correct the evolution near the Big-Bang (more precisely, when  $|T|$  is of the order of  $\lambda$ ). It was found that the Big-Bang is removed and replaced with an exponential expansion ( $H(t)$  goes to  $\sqrt{\lambda/12}$  when  $t \rightarrow -\infty$ ) for any state equation  $p = w\rho$  with  $w > -1$ . As a consequence, the particle horizon diverges and the whole universe turns out to be causally connected.

Other no less important issues, such as the growth of fluctuations,<sup>72–74</sup> the observational constraints<sup>75–77</sup> or the variation of the universal constants,<sup>78,79</sup> have also been studied in  $f(T)$  cosmology.

### B. Cosmic strings

Static circular<sup>80</sup> or spherically<sup>81,82</sup> symmetric solutions are also analyzed in the  $f(T)$  literature. In particular, it has been shown that the Schwarzschild geometry remains as a solution of  $f(T)$  theories.<sup>83</sup> The issue of removing singularities in stationary configurations was studied in a slightly different framework of modified TEGR, by using a Lagrangian density inspired in Born-Infeld electrodynamics:<sup>84</sup>

$$\begin{aligned} \mathcal{L} &= -\frac{\lambda}{2\kappa} \left[ \sqrt{\det[g_{\mu\nu} - 2\lambda^{-1} F_{\mu\nu}]} - \sqrt{-g} \right] \\ &\xrightarrow{\lambda \rightarrow \infty} \frac{1}{2\kappa} \sqrt{-g} \text{Tr}(F), \end{aligned} \quad (49)$$

( $g_{\mu\nu}$  is that of Eq. (39)). TEGR is recovered in the limit  $\lambda \rightarrow \infty$  if  $\text{Tr}(F) = \mathbb{S} \cdot \mathbb{T}$ . A possible choice, but not the only one, is  $F_{\mu\nu} = S_{\mu\lambda\rho} T_\nu^{\lambda\rho}$ ; then

$$\mathcal{L} = \frac{e}{2\kappa} \left[ \mathbb{S} \cdot \mathbb{T} - \frac{\lambda^{-1}}{2} (\mathbb{S} \cdot \mathbb{T})^2 + \lambda^{-1} F_\mu^\nu F_\nu^\mu \right] + \mathcal{O}(\lambda^{-2}). \quad (50)$$

This expression shows that the theory (49) modifies General Relativity at high curvatures and differs from a mere  $f(T)$  theory. These features make it potentially able of avoiding the singular Schwarzschild solution or any other solution having  $\mathbb{S} \cdot \mathbb{T} = 0$ .<sup>63,83</sup>

The Lagrangian (49) was used in Ref. 84 to heal the singular behavior of a cosmic string:

$$ds^2 = d(t + 4J\theta)^2 - Y^2(\rho) d\rho^2 - \rho^2 M^2 d\theta^2 - dz^2. \quad (51)$$

In General Relativity it is  $Y = 1$ . In particular, if the dimension is reduced to  $D = 2+1$  ( $z$  is removed), the  $Y = 1$  case is a solution of Einstein equations for  $T^{00} = \mu \delta(x, y)$  and  $T^{0i} = (J/2) \epsilon^{ij} \partial_j \delta(x, y)$ , where  $\mu \equiv (1 - M)/4$ . So the solution (51) looks like the geometry associated with a particle of mass  $\mu$  and spin  $J$  (a *cosmon*).<sup>85</sup> However, no gravitational field surrounds the cosmon since the metric is manifestly flat (in terms of the Levi-Civita curvature). Instead, the presence of a cosmon only produces topological effects: the deficit angle  $8\pi\mu$  (conical singularity), and the existence of closed timelike curves (CTC) of constant  $(t, \rho, z)$  when  $\rho < \rho_o \equiv 4J/M$ :

$$ds^2 = -\left(\rho^2 - \frac{16J^2}{M^2}\right) M^2 d\theta^2. \quad (52)$$

When the geometry (51) is treated within the modified gravity framework ruled by the Lagrangian (49), then  $Y$  becomes a  $J$ -depending function of  $\rho$ .  $Y(\rho)$  goes to 1 for  $\rho \gg 4J/M$  (GR limit) but diverges for  $\rho \rightarrow 4J/M$ .<sup>84,86</sup> Besides, the solution  $J = 0$  coincides with the respective GR solution. While  $J$  in General Relativity has no local effects ( $J$  could be locally absorbed through the coordinate change  $t' = t + 4J\theta$ ), now the integration constant  $J$  is a physically relevant degree of freedom that fixes the scale ruling the GR limit. The curved geometry that replaces the GR cosmic string has remarkable features: i)

the Levi-Civita curvature is well behaved at  $\rho_o = 4J/M$  ( $R$ ,  $R_{\mu\nu}R^{\mu\nu}$  and  $R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho}$  vanish at  $\rho_o$ ), ii) an infinite proper time is required to reach  $\rho_o$ , and iii) no CTC's are left. So, the theory (49) successfully smoothes the GR cosmic string.

### C. Degrees of freedom in $f(T)$ theories

$f(T)$  gravity is structurally simpler than metric  $f(R)$  theories, because it always produces second order dynamical equations. However, this nice feature does not prevent  $f(T)$  gravity from displaying additional degrees of freedom. The circular symmetric solution of the previous section, obtained in the context of an extension of  $f(T)$  gravity, exhibits a local degree of freedom associated with the integration constant  $J$  that is only globally apparent in the corresponding GR solution. Since  $f(T)$  is a theory not for the metric but for the tetrad, it contains more degrees of freedom than GR. In fact, many different tetrads lead to the same metric, since the relation (39) is invariant under local Lorentz transformation of the tetrad field. In order that a theory for tetrads has the same degrees of freedom than a theory for the metric, its action should be invariant under local Lorentz transformation of tetrads in the tangent space. TEGR is a particular case accomplishing this condition: although  $\mathbb{S} \cdot \mathbb{T}$  does vary under local Lorentz transformation of the tetrad field, the variation is located in the divergence term of Eq. (44); therefore the dynamics does not vary. But in a  $f(T)$  theory, the variation affects the dynamics because the divergence term remains encapsulated in the function  $f$ . Only a global Lorentz invariance survives in such case. Because of this reason, a  $f(T)$  theory globally determines the field of tetrads; it provides the spacetime with a global frame that fixes its metric and endows it with a *parallelization*. A local Lorentz transformation would destroy the parallelization (consider, for instance, a Cartesian grid in Minkowski spacetime). As was proven in Ref. 87, the local Lorentz invariance cannot be restored by adding the action with a spin connection. The issue of counting the number of degrees of freedom in a  $f(T)$  theory could be tackled by reformulating the  $f(T)$  action in a Brans-Dicke-like form.<sup>88</sup> Nevertheless, the counting the first and second class constraints in the Hamiltonian formulation shows that  $f(T)$  theories in four dimensions have five degrees of freedom.<sup>89</sup>

Summarizing, in passing from TEGR to  $f(T)$  gravity, we are replacing a local symmetry with a global one; in return, we would be converting global degrees of freedom (like the topological  $J$  in the cosmic string) into local degrees of freedom. These new local degrees of freedom could be essential to heal singularities. As a last remark, it should be realized that, even if the geometry is highly symmetric, it could be very hard to exploit the symmetry to anticipate aspects of the tetrad field parallelizing such a geometry. This causes that the naive diagonal choice we used in Eq. (47) does not work for open and closed

FRW universes.<sup>90</sup>

## VI. CONCLUSIONS

$f(R)$  and  $f(T)$  theories are alternative ways to modify General Relativity. Like metric  $f(R)$  theories,  $f(T)$  gravity contains additional degrees of freedom. However, these additional degrees of freedom do not appear as a consequence of the higher order of the dynamical equations, since  $f(T)$  gravity always leads to second order equations. They appear because  $f(T)$  gravity provides the spacetime not only with a metric but with a global parallelization. An extension of  $f(T)$  gravity –the one governed by the determinantal Lagrangian of Eq. (49)– shows that the extra degrees of freedom can play a fundamental role in smoothing singularities. In the case of the cosmic string, a global (topological) property of the GR solution, encoded in the constant  $J$ , becomes a local degree of freedom entering the metric tensor. This generates a family of geometries parametrized by  $J$ , which includes a GR solution as a particular case (the  $J = 0$  case).

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- <sup>92</sup>Notice that  $g_{00} \simeq 1 - \sqrt{2\kappa/3} \tilde{\phi}$ , because  $\sqrt{\kappa} |\tilde{\phi}| \ll 1$ . Therefore  $g_{00}$  implies a weak Yukawa-shaped gravitational potential  $\Phi(r) = -\sqrt{\kappa/6} \tilde{\phi}(r)$  with  $m R_\odot \ll 1$ . Because of Eq. (25),  $\Phi$  can be ignored, which is the realization of the chameleon effect.